Exercise sheet 12

For exercises 1 and 2, you may assume that there exists a constant $C > 0$ such that any zero ρ of $\zeta(s)$ satisfies
$$\Re(\rho) < 1 - \frac{C}{\log\left(2 + |\Im(\rho)|\right)}.$$
For exercises 5-7, you may assume that there exists a constant $C > 0$ such that if χ is a Dirichlet character modulo q then any zero ρ of $L(s, \chi)$ satisfies
$$\Re(\rho) < 1 - \frac{C}{\log\left(q(1 + |\Im(\rho)|)\right)},$$
unless the possibility of at most one real zero $1 - \frac{C}{\log q} < \beta < 1$ when χ is quadratic.

1. Let $s = \sigma + it$ with $\sigma > 1 - \frac{C}{2\log(2+|t|)}$ and $|t| \geq 2$. Then

(a) $\left|\frac{\zeta'}{\zeta}(s)\right| \ll \log(|t| + 2)$;

Hint: Show first for $s_1 = 1 + \frac{1}{\log(|t|+2)} + it$. Then show $\left|\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1)\right| \ll \log(|t| + 2)$ using that
$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1) = \sum_{\rho : |s-\rho| \leq 1} \left(\frac{1}{s-\rho} - \frac{1}{s_1 - \rho}\right) + O\left(\log(|t| + 2)\right)$$
and that $|s - \rho| \asymp |s_1 - \rho|$.

(b) $|\log \zeta(s)| \leq \log\log(|t| + 2) + O(1)$;

Hint: Use that for $u > 1$ we have $\zeta(u) < 1 + \frac{1}{u-1}$. Show again first for s_1 and use that $\log(s) - \log(s_1) = \int_{s_1}^{s} \frac{\zeta'}{\zeta}(z)dz$.

(c) $\left|\frac{1}{\zeta(s)}\right| \ll \log(|t| + 2)$.

Hint: $\log\left|\frac{1}{\zeta(s)}\right| = -\Re \log \zeta(s)$.-

2. (a) We can prove the Prime Number Theorem directly using Perron formula's without using the explicit formula. Use Perron's formula along with Cauchy's Residue Theorem to show that there exists a constant $c > 0$ such that
$$\psi(x) = x + O(xe^{-c\sqrt{\log x}}).$$
Hint: You may shift the contour from $\sigma = 1 + \frac{1}{\log x}$ to $\sigma = 1 - \frac{C}{2(\log T)}$ and choose T optimally. Use the bound for $\frac{\zeta'}{\zeta}$ from exercise 1.

(b) Let $M(x) = \sum_{n \leq x} \mu(n)$. Show that
$$M(x) \ll xe^{-c\sqrt{\log x}}.$$
Hint: Apply the same method for $\frac{1}{\zeta(s)}$.

(c) Show that
$$\sum_{n\le x}\frac{\mu(n)}{n} \ll e^{-c\sqrt{\log x}}.$$
Hence in particular deduce that
$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0.$$

3. (a) Show that there exists a constant $0<\theta<1/2$ such that $\zeta(s)$ has no zeros in the region $\mathfrak{Re}(\rho)>1-\theta$ if and only if for any $\epsilon>0$, $\psi(x)=x+O(x^{1-\theta+\epsilon})$.

(b) Thus show that
$$\sum_{n\le x}\mu(n)=O(x^{1-\theta+\epsilon})$$
if and only if
$$\psi(x)=x+O(x^{1-\theta+\epsilon}).$$

4. Show that if χ is a character modulo q and $3/4\le\sigma\le 2$, then
$$-\frac{L'}{L}(s,\chi)=\frac{\mathbf{1}_{\chi=\chi_0}}{s-1}-\sum_{\rho:|\rho-s|\le 1}\frac{1}{s-\rho}+O\left(\log(q(|t|+1))\right).$$

Hint: Show first for primitive characters.

5. Show that there exists a universal constant c_1 such that if $q\le\exp(2c_1\sqrt{\log x})$ and $L(s,\chi)$ has no exceptional zero, then
$$\psi(x,\chi)=\mathbf{1}_{\chi=\chi_0}x+O\left(x\exp(-c_1\sqrt{\log x})\right),$$
but if $L(s,\chi)$ has an exceptional zero β, then
$$\psi(x,\chi)=-\frac{x^{\beta}}{\beta}+O\left(x\exp(-c_1\sqrt{\log x})\right).$$

6. Let $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ two distinct, real, primitive characters.

(a) For $\sigma>1$, show that
$$-\frac{\zeta'}{\zeta}(\sigma)-\frac{L'}{L}(\sigma,\chi_1)-\frac{L'}{L}(\sigma,\chi_2)-\frac{L'}{L}(\sigma,\chi_1\chi_2)\ge 0.$$

(b) Let β_j be a real zero of $L(s,\chi_j)$, for $j=1,2$. Deduce that for $\sigma>1$,
$$\frac{1}{\sigma-\beta_1}+\frac{1}{\sigma-\beta_2}\le\frac{1}{\sigma-1}+O\left(\log(q_1q_2)\right).$$

(c) Show that there exists a constant $c_3>0$ such that $\min(\beta_1,\beta_2)<1-\frac{c_3}{\log(q_1q_2)}$.

(d) Show that there exists a constant $c_4>0$ such that for $Q\ge 3$, there exists at most one $q\le Q$ for which it exists a real primitive character $\chi \pmod q$ with a real zero $\rho>1-\frac{c_4}{\log Q}$.

7. Assume that for all $\epsilon>0$, there exists a constant $C(\epsilon)$ such that for all for each real, primitive character χ modulo q, we have $L(1,\chi)\ge C(\epsilon)q^{-\epsilon}$.

(a) Show that there exists $C'(\epsilon)$ such that for each χ modulo q, a real zero β of $L(s,\chi)$ satisfies $\beta \le 1 - C'(\epsilon)q^{-\epsilon}$.
Hint: You may use that $|L'(\sigma,\chi)| \ll (\log q)^2$, when $\sigma > 1 - \frac{C}{2\log q}$.

(b) Show that for any positive A, there exists a constant $c_2 = c_2(A)$ such that for all $q \le (\log x)^A$ and χ modulo q, we have

$$\psi(x,\chi) = \mathbf{1}_{\chi=\chi_0} x + O\left(x \exp(-c_2\sqrt{\log x})\right).$$

(c) Deduce that

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O\left(x \exp(-c_2\sqrt{\log x})\right)$$

and

$$\pi(x;q,a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O\left(x \exp(-c_2\sqrt{\log x})\right).$$

8. (a) Let z be a complex number such that $z \ne 0, -1, -2\dots$ Show that

$$\frac{\Gamma'}{\Gamma}(z) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right).$$

(b) Using the Euler–Maclaurin formula, or otherwise, deduce that if $|\arg(z)| < \pi$, then

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O\left(\frac{1}{(z+1)^2}\right).$$